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THE PROPAGATION OF CONVERGING AND DIVERGING SHOCK WAVES UNDER INTENSE HEAT-EXCHANGE CONDITIONS†

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The motion of a gas with strong diverging shock waves, generated by supplying energy to the gas in accordance with a power law, and with converging shock waves in the cases of spherical and cylindrical symmetry is investigated. It is suggested that the intense heat exchange ensures that the temperature is the same over the whole volume of the moving gas and equal to the temperature of the particles directly behind the shock wave, on which the laws of conservation of mass, momentum and energy are satisfied.

The adiabatic motion of a perfect gas with strong cylindrical and spherical converging shock waves has been investigated in [1–3]. In particular, it was established in [3] that for values of the adiabatic index $\gamma \in (1; 1.87)$, a solution of the self-similar problem with a spherical shock wave exists and is unique. When $\gamma > 1.87$ the uniqueness breaks down. The motion of a gas with a diverging shock wave has been investigated in detail. In particular, a solution of the problem of a high-power point explosion when there is intense heat exchange behind the shock wave is given in [4, 5].

Unlike [4, 5], in this paper we assume that all the conservation laws are satisfied on the shock wave, and under these conditions we investigate the effect of homothermal conditions on the flow parameters and the characteristics of the motion of converging shock waves and also diverging shock waves, generated by supplying heat to the gas in accordance with a power law.

1. THE PROBLEM OF A CONVERGING SHOCK WAVE IN A PERFECT GAS WHEN THERE IS INTENSE HEAT EXCHANGE

We will consider the problem of the propagation of converging spherical and cylindrical shock waves in a perfect gas in the case of homothermal flow.

Idealizing the actual process, we will assume that energy is supplied to an unbounded volume of gas at infinity, as a result of which a strong shock wave is formed (we will neglect the back pressure); we will denote the required dependence of the radius of this shock wave on the time t by $R(t)$. We will assume that the shock wave propagates in a stationary gas with constant initial density ρ_0; behind the shock wave the gas flow is continuous and homogeneous.

Since the flow is homothermal the gas temperature T behind the shock wave depends only on time. This model describes a gas with infinitely high thermal conductivity so that the temperature in the medium becomes equalized after a negligibly short time. The velocity of motion of the gas has only a radial component u and it, like the pressure p and the density ρ, depends only on time and the radial coordinate r in a spherical system of coordinates (or, respectively, a cylindrical system of coordinates).

In addition we will assume that the shock wave reaches a centre of symmetry at the instant $t = 0$, i.e. instants of time $t < 0$ correspond to the motion of the gas before it is focused.

Hence, the required functions $\rho(r, t)$, $u(\mathbf{r}, t)$, $T(t)$ and $p(r, t)$ in the region $\{(r, t): t < 0; r > R(t)\}$ are described by the following system of equations

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\rho\frac{\partial u}{\partial r}+\frac{(\nu-1)\rho u}{r}=0,\quad \frac{\partial T}{\partial r}=0$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{R_0T}{\rho}\frac{\partial\rho}{\partial r}=0,\quad p=\rho R_0T \tag{1.1}$$

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where R_0 is the gas constant, and ν is the dimension of space ($\nu = 3$ for spherical shock waves and $\nu = 2$ for cylindrical shock waves). The relations at the pressure jump have the form

$$u(R(t), t) = \frac{2}{\gamma+1} D(t), \quad \rho(R(t), t) = \frac{\gamma+1}{\gamma-1}\rho_0, \quad T(t) = \frac{2(\gamma-1)}{(\gamma+1)^2 R_0} D^2(t) \tag{1.2}$$

where $D(t) = \dot{R}(t)$ is the velocity of the shock wave and $\gamma > 1$ is the Poisson adiabatic index.

The problem is self-similar in this formulation [6], and its solution can naturally be sought in the form

$$R(t) = c(-t)^n, \; n > 0; \; u(r, t) = \sqrt{\theta_2} D(t) f(\xi)$$
$$\rho(r, t) = \rho_0 g(\xi), \; T(t) = \theta_2 D^2(t) / R_0; \; \xi = r / (\sqrt{\theta_2} R(t)) \tag{1.3}$$

with unknown self-similarity factor n; c and θ_2 are unknown positive constants. Substituting (1.3) into (1.1) we obtain the following system of equations

$$f' = \frac{g'}{g}(\xi - f) - (\nu - 1)\frac{f}{\xi}, \quad \frac{g'}{g} = f'(\xi - f) + \left(\frac{1}{n} - 1\right) f, \quad p = \rho_0 g(\xi)\theta_2 D^2(t) \tag{1.4}$$

where $\xi \in (\xi_0; \infty)$, $\xi_0 = 1/\sqrt{(\theta_2)} > 0$. In the new variables, the relations on the shock wave (where $\xi = \xi_0$) take the form

$$f(\xi_0) = \sqrt{\frac{2}{\gamma-1}} = f_0, \quad g(\xi_0) = \frac{\gamma+1}{\gamma-1}, \quad \theta_2 = \frac{2(\gamma-1)}{(\gamma+1)^2} \tag{1.5}$$

It can be seen that it is sufficient to determine the value of n and the function f, which are in the half-space $[\xi_0; \infty)$, by solving the Cauchy problem

$$f' = \frac{f[(\xi - f)\xi(1/n - 1) - (\nu - 1)]}{\xi[1 - (\xi - f)^2]} = F(\xi, f, n), \quad f(\xi_0) = f_0 \tag{1.6}$$

The function g is then defined by the formula

$$g(\xi) = \frac{\gamma+1}{\gamma-1}\exp\left\{-\frac{f^2(\xi)}{2} + \xi f(\xi) - \frac{\gamma}{\gamma-1} + \frac{1-2n}{n}\int_{\xi_0}^{\xi} f(x)dx\right\} \tag{1.7}$$

which solves the problem apart from a positive constant c, the value of which depends on the method by which the energy is introduced.

Hence, the problem in question reduces to determining the parameter n and the solution of problem (1.6) for this value of the parameter. Here there are additional conditions imposed on the required solution f, defined in the half-space $[\xi_0; \infty)$:

1. since, at the instant when focusing occurs, the velocity of motion behind the shock wave must be finite, we have $\xi^{1/n-1} f(\xi) \to O$ as $\xi \to \infty$, where C is a certain constant;

2. since, at any fixed instant of time $t < 0$, the modulus of the velocity of the gas particles should not increase with distance from the shock wave, by virtue of (1.3) we have $d|f|/d\xi \leq 0$ when $\xi \in (\xi_0; \infty)$.

We will show that when $\gamma \in (1; 3)$ these additional conditions uniquely define the values of the parameter n and, consequently, all the characteristics of the motion considered; when $\gamma \geq 3$ the self-similar parameter n can take any values in the interval $(0; (\gamma + 1)/(\gamma + 1 + 2(\nu - 1))]$.

For an arbitrary value of the self-similar parameter $n > 0$ and for $\nu = 2, 3$, we will consider the integral curve of Eq. (1.6) emerging from the point $(\xi_0, f_0) = ((\gamma + 1)/\sqrt{(2(\gamma - 1))}, \sqrt{(2/(\gamma - 1))})$ and we will determine the values of the parameter n for which, corresponding to the values $\xi \geq \xi_0$, part of this integral curve is a graph of the function $f = f(\xi)$ defined on the semiaxis $\xi \geq \xi_0$ and satisfying conditions 1 and 2.

Since for all values of the parameters the coordinates of the initial point (ξ_0, f_0) satisfy the inequalities $\xi_0 > 1, f_0 > 0$, and the ray $f = 0, \xi > 1$ is an integral curve on which there are no singular points, the integral curve of interest to us when $\xi \geq \xi_0$ lies in the quarter $\{\xi \geq \xi_0, f \geq 0\}$ of the phase plane.

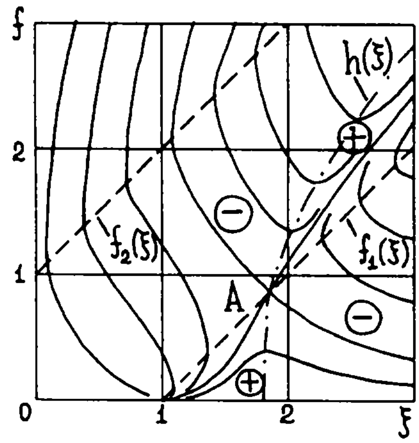

Fig. 1.

Note that in this quarter there is a singular point only when $\xi_0/(\nu - 1 + \xi_0) \leqslant n < 1$ and it is unique when $1 < n \leqslant \xi_0/(\xi_0 - (\nu - 1))$ we will denote it by A. In the first case it is a saddle and has coordinates $(n(\nu - 1)/(1 - n), (n\nu - 1)/(1 - n))$ (it lies on the straight line $f = \xi - 1$ and its coordinates are a monotonically increasing function of n).

An integral curve which satisfies these conditions does not exist for $n \geqslant 1$.

In fact, in this situation the right-hand side of Eq. (1.6) is positive for all $0 < f < \xi - 1$. By virtue of condition 1, for sufficiently large values of ξ, the required integral curve must lie below the straight line $f = \xi - 1$, and, consequently, we cannot have $f(\xi) > 0$ on it, which contradicts condition 2.

We will now consider the case when $n \in (0; 1)$. In this situation the right-hand side F of Eq. (1.6) is negative at points in the first quarter, which lies to the right of the straight line $f = \xi - 1 = f_1(\xi)$ and of the curve $f = \xi - (\nu - 1)/(\xi(1/n - 1)) = h(\xi)$ (on this curve the vector field given by Eq. (1.6) is horizontal). Between these lines $F(\xi, f, n) > 0$ and, by virtue of condition 2, the integral curve of interest to us cannot lie in this zone. To the left of these lines and to the right of the straight line $f = \xi + 1 = f_2(\xi)$ the sign of the right-hand side of (1.6) is again negative (Fig. 1). We can also obtain from (1.6) that along any integral curve in the region $0 < f < \min\{\xi - 1; h(\xi)\}$ $f(\xi) \sim C\xi^{(n-1)/n}$ as $\xi \to \infty$ with a certain positive constant C, $f(\xi)$ satisfies condition 1.

The initial point (ξ_0, f_0) lies on the curve $\xi = f + 1/f$, and when $1 < \gamma < 3$ it lies above the straight line $f = \xi - 1$, when $\gamma > 3$ it lies below this straight line, and when $\gamma = 3$ it is the point at which they intersect.

Suppose $1 < \gamma < 3$. Then the integral curve of interest to us must obviously pass through the singular point. Consequently, $n \in [\xi_0/(\nu - 1 + \xi_0); 1)$, the singular point is a saddle, and the integral curve considered is a separatrice.

For any fixed $\gamma \in (1; 3)$ and $\nu = 2; 3$ there is also a unique value of n (for which the integral curve (1.6), emerging from the point (ξ_0, f_0), is the separatrice of the saddle A considered, the coordinates of which we will denote by $(\xi_A(n), f_A(n))$. Note that since the coordinate $f_A(n)$ must not be greater than f_0, it makes sense to consider only the following values of the parameter n

$$n_1 = \frac{\xi_0}{\nu - 1 + \xi_0} \leqslant n \leqslant \frac{f_0 + 1}{\nu + f_0} = n_2 \quad (n_1 < n_2) \tag{1.8}$$

We will first prove that the required value of the parameter n is unique. We will assume that two different values of n, n_* and n_{**}, exist (we assume that $n_* > n_{**}$), for which the separatrice of the saddle A with a negative angle of inclination of the tangent emerges from a single point (ξ_0, f_0). We will consider the set of points of intersection of these separatrices (for $n = n_*$ and for $n = n_{**}$); they all, obviously, lie in the triangle $\Delta = \{\xi - 1 < f \leqslant f_0, \xi \geqslant \xi_0\}$. We will denote the point of intersection which has the greatest abscissa by (ξ_*, f_*). It can be shown that such a point exists.

Since $\xi_A(n_*) > \xi_A(n_{**})$, a comparison of the angular coefficients of the slope of the tangents at the point (ξ_*, f_*) to these separatrices gives the inequality $F(\xi_*, f_*, n_*) \geqslant F(\xi_*, f_*, n_{**})$, which contradicts the fact that $\partial F/\partial n < 0$ in the whole of the triangle Δ, and hence at the point (ξ_*, f_*). The contradiction obtained proves that the required value of n is unique.

Note that we have in fact proved a somewhat stronger assertion: the separatrices of the saddle A with negative angle of inclination of the tangent corresponding to different values of the parameter $n \in [n_1; n_2]$ cannot intersect in the strip $\xi > f > \xi - 1$.

We will now prove that the required value of n exists.

We will assume that no value of n exists for which the integral curve of Eq. (1.6), passing through the point (ξ_0, f_0), is a separatrice of the saddle A with negative slope of the tangent. Then, for any value of n which satisfies inequalities (1.8), the integral curve emerging from the point (ξ_0, f_0) when $\xi > \xi_0$ either passes above the separatrice of the saddle considered (the set of values of these n will be denoted by N_+), or below it ($n \in N_-$); $[n_1; n_2] \subset \{N_+ \cup N_-\}$ and $N_+ \cap N_- = \varnothing$. Since when $n = n_1$ we have $\xi_A(n_1) = \xi_0$, $af_A(n_1) < f_0 = f(\xi_0)$, then $n_1 \in N_+$. For $n = n_2$ we have $\xi_A(n_2) > \xi_0$, while $f_A(n_2) = f_0 = f(\xi_0)$; hence $n_2 \in N_-$. Consequently, the sets N_+ and N_- are not empty.

By virtue of the fact that the right-hand side of (1.6) depends continuously on n it can be shown that if a certain value of n belongs to the set N_+, a certain neighbourhood of this value exists which also belongs to the set considered; similarly any $n \in N_-$ belongs to the set N_- together with some of its neighbourhood. Hence, we obtain that the interval $[n_1; n_2]$ is covered by two non-empty open non-intersecting sets, which is impossible. Thus, we have proved that the required value of n exists.

If $\gamma = 3$, then when $0 < n < 2/(\nu + 1)$ the point (ξ_0, f_0) lies on the straight line $f = \xi - 1$ above the point A and the branch of the integral curve, which lies in the region $0 < f < \xi - 1$, is the required integral curve. When $n = 2/(\nu + 1)$ the point (ξ_0, f_0) coincides with the point A. The function, the graph of which contains the separatrice which passes through the region $0 < f < \xi - 1$ is the required relationship f. When $n \in (2/(\nu + 1); 1)$ the function $f(\xi)$ along the integral curve emerging from the point (ξ_0, f_0), as can easily be seen, does not satisfy condition 2.

Suppose now that $\gamma > 3$. Then if $n \in (0; (\gamma + 1)/(\gamma + 1 + 2(\nu - 1))]$, the point (ξ_0, f_0) lies in the region where $0 < f \leqslant h(\xi)$, and the integral curve emerging from it obviously satisfies the required conditions. If $(\gamma + 1)/(\gamma + 1 + 2(\nu - 1)) < n < 1$, the point (ξ_0, f_0) falls in the region $f > h(\xi)$ and condition 2 is not satisfied.

Thus we have proved that for any values of $1 < \gamma < 3$ and $\nu = 2; 3$, a self-similar solution of the problem which satisfies the imposed limitations exists and is unique. It can be shown that when γ increases the value of the required self-similar parameter decreases.

For certain γ the values of n can be determined numerically. The results of a calculation are shown in Table 1. For comparison we give in parentheses the self-similarity factors in the case of adiabatic gas flow [7]. When $\gamma \geqslant 3$ the problem has an infinite set of self-similar solutions. Note that in the case of adiabatic flow with spherical symmetry [3] the "threshold" value of γ, after which the uniqueness breaks down, is equal to 1.87.

Note that by virtue of the limitations imposed, all possible values of n are less than unity, and the functions $f(\xi)$ corresponding to them satisfy the condition

$$f(\xi) \sim C\xi^{-(1-n)/n} \quad \text{as} \quad \xi \to \infty \tag{1.9}$$

with a certain positive constant C. Then, using (1.3), we obtain that at any fixed instant of time $t \neq 0$ in the region of gas flow behind the shock wave, the velocity modulus tends to zero as $r \to \infty$ proportional to $r^{-(1-n)/m}$.

The function $g(\xi)$ (from (1.4)) satisfies the equation

$$\frac{g'}{g} = \frac{f[(1/n - 1)\xi - (\nu - 1)(\xi - f)]}{\xi[1 - (\xi - f)^2]} \tag{1.10}$$

Using the properties of the functions $f(\xi)$, we can obtain that when $1 < \gamma < 3$ the function $g(\xi)$ is a monotonically increasing function in the interval $[\xi_0; \infty)$. In this case, at any fixed instant of time in the region of gas flow behind the shock wave, the density and, of course, the pressure, are increasing functions. When $\gamma \geqslant 3$, when $n \in (0; 1/\nu]$, at any fixed instant of time the density and the pressure are

Table 1

γ	$\nu = 2$	$\nu = 3$
5/3	0.722 (0.816)	0.559 (0.688)
1.4	0.753 (0.835)	0.597 (0.717)
1.2	0.797 (0.861)	0.656 (0.757)

Table 2

γ	$\nu = 2$	$\nu = 3$
5/3	5.9	7.3 (9.55)
1.4	10.2	13.5 (20.1)
1.2	23.2	35.2

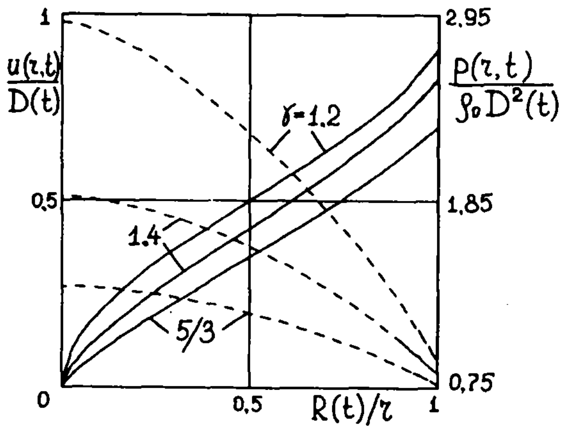

Fig. 2.

monotonically decreasing functions of the coordinate r; when $n \in [(\gamma + 1)/(\nu(\gamma - 1) + 2); (\gamma + 1)/(\gamma + 1 + 2(\nu - 1))]$ they are monotonically increasing functions of the spatial coordinate; when $n \in (1/\nu; (\gamma + 1)/(\nu(\gamma - 1) + 2))$ the characteristics considered first decrease and then increase as r increases.

Graphs of $u(r, t)/D(t)$ (the continuous curves) and $p(r, t)/(\rho_0 D^2(t))$ (the dashed curves) as a function of $R(t)/r$ at a fixed instant of time in the case of flow with spherical symmetry are shown in Fig. 2 for certain values of γ.

To determine the behaviour of $g(\xi)$ as $\xi \to \infty$, using (1.9) and (1.10) we obtain the equation

$$\frac{g'}{g} = -\left(\frac{1-n}{n} - (\nu - 1)\right)\frac{C}{\xi^{1+1/n}}$$

It can be seen that $g(\xi) \to g_{np} \geqslant 0$ as $\xi \to \infty$.

At the instant when focusing occurs, when $t = 0$, the following pattern is observed: $R(t) = 0$, and consequently, from (1.3) when $r > 0$, we have $\xi = \infty$; $u(r, 0) = Cr^{-(1-n)/n}$, where C is a certain constant and $\rho(r, 0) = \rho_{np}$. The values of the ratio ρ_{np}/ρ_0 are shown in Table 2 for certain values of γ. In parentheses we give the values of ρ_{np}/ρ_0 for adiabatic gas flow [8]. It can be seen that, in the case considered, the medium is compressed less than in the adiabatic process.

The condition at the discontinuity is satisfied at the centre of the symmetry. By virtue of the fact that only values of $n < 1$ are possible, we have $D(0) = \infty$. Then, from (1.2) we have $u(0, 0) = \infty$ and $T(0) = \infty$. Consequently, at the instant when focusing occurs the pressure in the space is infinite due to the infinitely large increase in temperature. Focusing of the converging shock wave, taking into account radiant heat exchange, was considered qualitatively in [9], where it was found that the cumulation of energy at the instant when focusing occurs is unlimited due to the infinitely large increase in the density, while the temperature remains finite in this case.

We can solve this problem using the Chester–Chisnell–Whitham approximate method [7]. Using this, for the homothermal gas motion behind a shock wave of infinitely high intensity in a channel with a variable cross-sectional area $A(r)$, we can obtain an equation for the shock-wave velocity

$$\frac{k(\gamma)}{D}\frac{dD}{dr} + \frac{1}{A}\frac{dA}{dr} = 0, \quad k(\gamma) = 1 + \sqrt{\frac{2}{\gamma - 1}}$$

Hence it follows that $D = CA^{-1/k(\gamma)}$, where C is a constant.

We will apply the result obtained to converging spherical and cylindrical shock waves, taking $A(r) = Cr^2$ and $A(r) = Cr$, respectively, where C is an arbitrary constant.

We obtain that the velocity D is proportional to $r^{-(\nu-1)/k(\gamma)}$. For the self-similar solution it can be seen from (1.3) that the velocity D is proportional to $r^{-(1-n)/n}$. In Table 3 we give, for comparison, the exponents $(1 - n)/n$ and $(\nu - 1)/k(\gamma)$ for some values of γ. The relative value of the error between the approximate and accurate values of the exponent does not exceed 0.0855, i.e. the approximate method encapsulates the main feature of the flow.

Hence, a converging shock wave, in the case of homothermal motion of a gas, reacts primarily to the varying geometry of the flow. The other perturbations have a comparatively small effect.

Table 3

γ	$\nu = 2$		$\nu = 3$	
	$1/n - 1$	$1/k$	$1/n - 1$	$2/k$
5/3	0.385	0.366	0.789	0.732
1.4	0.328	0.309	0.675	0.618
1.2	0.255	0.240	0.525	0.481

2. UNSTEADY MOTION OF A PERFECT GAS UNDER CONDITIONS OF INTENSE HEAT EXCHANGE WHEN ENERGY IS SUPPLIED IN ACCORDANCE WITH A POWER LAW

We will consider an unbounded volume of a stationary perfect gas with constant initial density ρ_0 and constant pressure p_0, to which, beginning at a certain instant of time $t = 0$, energy is supplied in accordance with the law $E(t) = E_0t^k$, where $E(t)$ is the total amount of energy supplied in the time interval $[0; t]$, E_0 is a fixed positive number and $k = \text{const} > 0$. The power law of the supply of energy is achieved by appropriate motion of a spherical piston expanding from a centre of symmetry, the required dependence of the radius of which on time will be denoted by $r_p(t)$, i.e.

$$r_p(t)|_{t=0} = 0, \quad dr_p(t)/dt \geqslant 0 \quad \text{for} \quad t > 0$$

As a result of this, an intense shock wave propagates through the unperturbed gas. The required dependence of the radius of the shock wave on time will be denoted by $R(t)$ (the initial pressure p_0 will be neglected, i.e. $p_0 = 0$).

We will assume that the gas flow between the piston and the shock wave is continuous and one-dimensional. Since the conditions are homothermal, the temperature T in the region considered depends only on time.

The required functions, namely, the density $\rho(r, t)$, the radial component of the velocity $u(r, t)$, the temperature $T(t)$ and the pressure $p(r, t)$, the radial component of the velocity $u(r, t)$, the temperature $T(t)$ and the pressure $p(r, t)$ in the region $\{(r, t) : t > 0, r_p(t) < r < R(t)\}$ satisfy Eqs (1.1) for $\nu = 3$, the no-flow condition on the piston (for $r = r_p(t)$), and relations (1.2) on the shock wave.

Note that the homothermal motion of the gas, displaced by the piston, which is expanding in accordance with a power law, has been investigated in the case when the law of conservation of energy is satisfied over the whole of space and is not satisfied on the shock wave [10]. The motion of the piston is such that for any instant of time t the work done by it in the time interval $[0; t]$ is equal to E_0t^k. Then, taking the boundary condition on the piston into account, we obtain

$$4\pi \int_0^t p(r_p(x), x)u(r_p(x), x)r_p^2(x)dx = E_0t^k \tag{2.1}$$

We will take E^0 to be such that $E_0 = \alpha E^0$, where the value of α is chosen so that the dependence of the shock-wave radius on time is given by the formula

$$R(t) = (E^0 / \rho_0)^{1/5} t^{(2+k)/5}$$

Since the problem is self-similar, the solution will be sought in the form

$$\begin{gathered} r_p(t) = \lambda_p R(t), \quad u(r, t) = \sqrt{\theta_2} D(t) f(\xi), \quad \rho(r, t) = \rho_0 g(\xi) \\ T(t) = \theta_2 D^2(t) / R_0, \quad p(r, t) = \rho_0 D^2(t) \psi(\xi); \quad \xi = r / (\sqrt{\theta_2} R(t)) \end{gathered} \tag{2.2}$$

where $f(\xi)$, $g(\xi)$, $\psi(\xi)$ are unknown functions, λ_p and θ_2 are certain positive constants, and $D(t) = \dot{R}(t)$ is the shock wave propagation velocity.

Substituting (2.2) into (1.1) we obtain the following system of equations

$$f' = \frac{g'}{g}(\xi - f) - 2\frac{f}{\xi}, \quad \frac{g'}{g} = f'(\xi - f) + \frac{3-k}{2+k}f, \quad \psi(\xi) = g(\xi)\theta_2 \tag{2.3}$$

where $\xi \in (\xi_p, \xi_0)$, $\xi_p = \lambda_p/\sqrt{(\theta_2)}$, $\xi_0 = 1/\sqrt{(\theta_2)}$.

On the shock wave (with $\xi = \xi_0$) we have relations (1.5). The condition on the piston (with $\xi = \xi_p$) takes the form

$$f(\xi_p) = \xi_p \tag{2.4}$$

Using (2.3) and (2.4), relation (2.1) can be rewritten in the following form in the new variables

$$\alpha = 4\pi\xi_p^3\theta_2^{5/2}\left(\frac{2+k}{5}\right)^3 g(\xi_p)\frac{1}{k} \tag{2.5}$$

Hence, it is sufficient to determine the function $f(\xi)$, which is the solution of the Cauchy problem (1.6) with $n = (2+k)/5$, $\nu = 3$ (problem A) in the section $[\xi_p; \xi_0]$ ($\xi_p \neq 0$ satisfies Eq. (2.4)). The function $g(\xi)$ is then given by (1.7) (with $n = (2+k)/5$), which solves the problem formulated above uniquely.

Hence, the problem considered has been reduced to determining the function f, which is the solution of problem A in the range $[\xi_p; \xi_0]$, where the positive quantity ξ_p must satisfy condition (2.4).

By analysing the integral curve of the differential equation of problem A, it can be shown that for any value of the parameter $k > 0$ and any $\gamma \in (1; 5/3]$, a solution of the problem exists and is unique. For these values of the parameter γ the integral curve emerging from the point (ξ_0, f_0) for $\xi \in [\xi_p; \xi_0]$ lies in the first quarter in that part of the region $\xi - 1 < f < \xi$ where the right-hand side of the equation of problem A is less than zero. Hence, at any fixed instant of time t the velocity is a monotonically decreasing function of the coordinate r. In addition, it can be shown that for any fixed γ in the interval considered, at an arbitrary instant of time t

$$\frac{r_p(t, k_1)}{R(t, k_1)} \leqslant \frac{r_p(t, k_2)}{R(t, k_2)} \quad \text{for} \quad k_2 > k_1$$

The function $g(\xi)$ (from (2.3)) satisfies the equation

$$\frac{g'}{g} = \frac{f[-2(\xi - f) + (3-k)\xi/(2+k)]}{\xi[1 - (\xi - f)^2]}$$

using which we can obtain from (2.2) that for any fixed $\gamma \in (1; 5/3]$ with $k \geqslant 3$ at an arbitrary fixed instant of time $t > 0$, the density and the pressure are decreasing functions of the coordinate r, when $0 < k \leqslant (4f_0 - \xi_0)/(3\xi_0 - 2f_0)$ they are increasing functions, and for the remaining values of k the density and pressure reach their highest value inside the flow region between the piston and the shock wave. The temperature T of the gas moving behind the shock wave is proportional to $t^{2(k-3)/5}$.

The problem was solved numerically with $\gamma = 1.4$. In Fig. 3 we show graphs of $u(r, t)/D(t)$ (the continuous curves) and $\rho(r, t)/\rho_0$ (the dashed curves) as a function of $r/R(t)$ for different values of k.

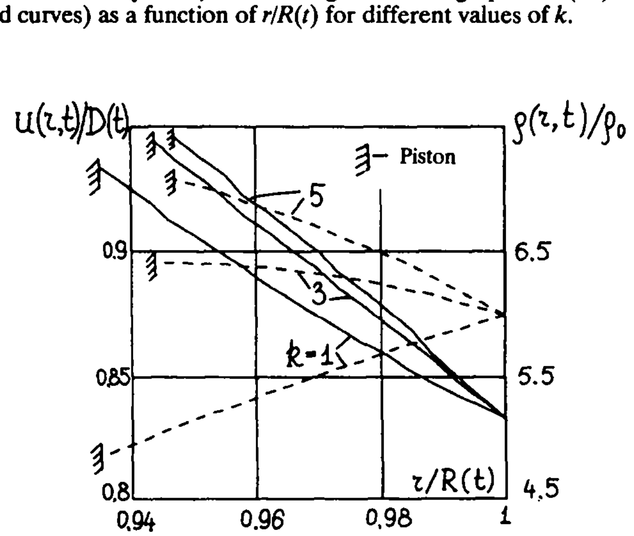

Fig. 3.

From the law of conservation of energy we can determine the amount of energy lost in the time interval $[0; t]$:

$$q(t,k) = E_0 t^k - 4\pi \int_{r_p(t,k)}^{R(t,k)} \left(\frac{\rho u^2}{2} + \frac{p}{\gamma - 1} \right) r^2 dr$$

We determine the values of the following quantities by numerical calculation

$$\tilde{r}_p = \frac{r_p(t)}{r_0(t)}, \quad \tilde{R} = \frac{R(t)}{r_0(t)}, \quad \tilde{q} = \frac{q(t,k)}{E_0 t^k}$$

where $r_0(t) = (E_0/\rho_0)^{1/5} t^{(2+k)/5}$ ($\tilde{r}_p$ and $\tilde{R}$ are the dimensionless radii of the piston and the shock wave) for some values of k. The results are given below.

k	0	1	3	3.154	5
$\tilde{R}$	0.000	0.924	0.796	0.787	0.705
$\tilde{r}_p$	0.000	0.863	0.751	0.743	0.667
$\tilde{q}$	1.000	0.258	0.008	0.000	−0.633

It turns out that a value of $k_* = 3.15408$ exists such that when $k = k_*$ the gas flow is adiabatic ($q(t, k_*) = 0$. When $k < k_*$ energy is radiated ($q(t,k) > 0$) outside the region considered; in the limiting case when $k \to 0$ the piston will not move, and all the energy supplied to the gas is radiated. When $k > k_*$ at any instant of time the energy of the gas will be greater than the total energy supplied to the system up to this time. This process is only possible when there is an additional supply of energy from outside ($q(t,k) < 0$).

Hence, the solution of the problem considered previously [4] of an abrupt point explosion ($E(t) = E_0$) is not the limiting solution for the family of solutions obtained.

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